United World College of the Atlantic

EXTENDED ESSAY: Mathematics

TITLE: *CAN THE WHOLE FIELD GROW INTO WEED?*

NAME: Jovana Šljivančanin

SESSION: May 2007

Abstract

In my extended essay I analysed, solved and extended a problem, which appeared in a Mathematical Olympiad. The text of the problem is as follows:
A piece of land of a square shape with dimensions 10m x 10m is divided into 100 square parcels with dimensions 1m x 1m. Initially, 9 of the parcels are overgrown by weed. If a parcel is surrounded by at least 2 parcels with weed from its sides after some time that parcel will be overgrown by weed. Can the whole piece of land grow into weed after some time?

In order to answer the question, I firstly analyzed some equivalent, simpler versions of the same problem and tried to solve them by system of exhaustion. This allowed me to construct my hypothesis: that with the given conditions the whole piece of land cannot grow into weed. The hypothesis was then proved by the system of following of change in a certain variable.
After the hypothesis was proved, the problem itself was generalised and it was shown that the hypothesis holds for even more complex, equivalent cases, when having a field with dimensions n x n and n-1 parcels with weed. Furthermore, a general condition necessary for the whole piece of land to grow into weed was found.

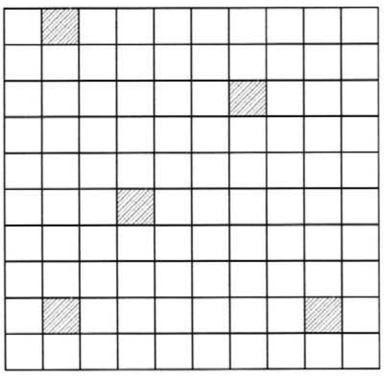
In the second part of the essay, this necessary condition was used to extend the initial problem. The extension was about finding all the possible arrangements of 10 parcels with weed which would result in the whole piece of land growing into weed.
Rather than drawing all the solutions, the system for finding all of the arrangements was developed by principle of recursion and backtracking. The system was then generalized for the piece of land with dimensions n x n and n weed parcels.

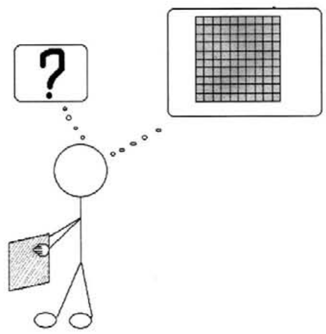
Contents

Introduction

The problem I am solving originally appeared in one of the Mathematical Olympiads in the1990s. The problem was found on the website:
http://www.matf.bg.ac.yu/~matic/competitions/dodatne/ZKOMB_mll.pdf
The text of the problem is:

A piece of land of a square shape with dimensions 10*m* x 10*m* is divided into 100 square parcels with dimensions 1*m* x 1*m*. Initially, 9 of the parcels are overgrown by weed. If a parcel is surrounded by at least 2 parcels with weed from its sides after some time that parcel will be overgrown by weed. Can the whole piece of land grow into weed after some time?





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Approaching the problem

In order to solve this problem, I will make use of the indirect approach to it. Namely, since the number of all the possibilities to arrange 9 parcels with weed in the 10x10 field is the number of ways of choosing 9 parcels out of 100 parcels:

$$^{100}C_9 = \frac{100!}{91!*9!} = 1902231808400,$$

it is obvious that an attempt to draw all the possible arrangements does not lead to the solution (unless one lives 3619162.497 years and draws one arrangement each minute, without having any breaks). Therefore, the way of solving this problem will involve finding a pattern that will enable us to arrange the parcels with the weed in a certain way, which will eventually lead to the whole piece of land growing into weed after some time. Failing to find such arrangement would mean that under the given conditions the whole piece of land can never grow into weed. In the further parts of the solution the parcels which are overgrown by weed will be referred to as *weed parcels* and the given piece of land will be referred to as *field*. Also, before I start solving the problem, I will introduce a definition and a theorem.

Definition: Two parcels will be called *adjacent* if and only if they have one common side.

Now the condition of the problem can be restated as: a parcel can become a weed parcel only if it has at least two adjacent weed parcels.

Theorem: If an arrangement of the weed parcels within the field can be obtained by rotating another arrangement of weed parcels through the angle $\boldsymbol{\alpha}$, the centre of rotation being any point **X**,($\mathcal{R}_{\mathbf{X},\,\boldsymbol{\alpha}}$), those two arrangements will produce the same number of new weed parcels.

proof: Let **F** be a field with any arrangement of the weed parcels. Let **F'** be a field obtained by rotating the field **F** about any point **X** through angle $\boldsymbol{\alpha}$. Therefore, the

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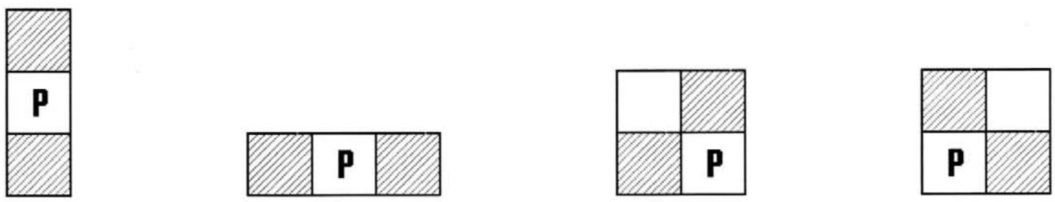
arrangement of the weed parcels in the field **F'** is obtained by rotating the arrangement of the weed parcels in the field **F** through the angle α. Since the number of parcels adjacent to any weed parcel **P** in field **F** is equal to the number of the weed parcels adjacent to parcel **P'** in field **F'**, where **P'** is obtained by rotating **P** about point **X** through the angle α, the number of the parcels which will turn into weed in the field **F'** is the same as number of the parcels which will turn into weed in the field **F**. The choice of the point **X** does not affect the result.

In order to find a solution to the problem, some equivalent simpler situations can first be analyzed. Since originally the field has dimensions 10*m* x 10*m* and 9 weed parcels, the equivalent cases would be having fields of dimensions n x n and n-1 weed parcels.
The simplest situation is having a field of dimensions 1*m* x 1*m*, with no weed parcels. Obviously, it is impossible for this field to grow into weed. It is also trivial to conclude that there are no solutions for the case when having a field of dimensions 2*m* x 2*m* and 1 weed parcel in it.

Field of dimensions 3*m* x 3*m*

Let's now analyse the case of having a field of dimensions 3*m* x 3*m* and 2 weed parcels in it. Let **P** be any parcel without weed in the field. The parcel **P** can become overgrown by weed only if the two given weed parcels are initially positioned so that they are adjacent to it. The possibilities of positioning the weed parcels in such way are shown in the diagrams below:

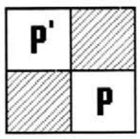


a) b) c) d)

Diagrams *b)* and *d)* can be obtained by rotating diagrams *a)* and *c)* about their centers through the angle of 90 degrees, respectively. Therefore, according to the theorem, arrangements *a)* and *b)* and arrangements *c)* and *d)* will produce the same number of weed parcels. Thus it is enough to examine only one of them in each of the equivalent pairs. Here, cases *a)* and *c)* will be examined.

<u>Case a):</u> In this way of positioning the weed parcels, only the parcel **P** will grow into weed. No matter where in the field we place the two weed parcels as in diagram *a)* no other parcels will have two weed parcels adjacent. Therefore, in this case it is impossible for the whole field to become overgrown by weed.

<u>Case c):</u> If the weed parcels are arranged like in the diagram *c)*, no matter where they are placed in the field, only parcels **P** and **P'** (look at the diagram below) will become weed parcels. None of the remaining 5 parcels will have two or more adjacent weed parcels, and thus it is impossible for the whole field to grow into weed.



Since in both of the cases *a)* and *c)* there are some parcels which can never grow into weed, the following conclusion can be drawn: *The whole field of dimensions 3m x 3m, initially with only 2 weed parcels, can not grow into weed.*

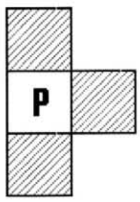
However, it is possible to find a maximum number of the parcels which will turn into weed. The maximum number of weed parcels which can be produced in the case of having a 3*m* x 3*m* field and 2 weed parcels is obtained in case c) and it is 4. So, at least 5 parcels in this field will never turn into weed.

Field of dimensions 4*m* x 4*m*

In order to try to make a hypothesis for the field of dimensions 10*m* x 10*m* with 9 weed parcels, I will examine one more equivalent simpler case. Namely, I will analyze the problem when having a field of dimensions 4*m* x 4*m* with 3 weed parcels at the beginning. A randomly chosen parcel **P** without weed can thus become a weed parcel:

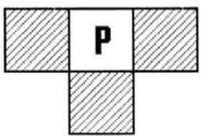
- if it is adjacent to all three weed parcels; or
- if it is adjacent to two weed parcels.

1st case: Parcel **P** is adjacent to all three weed parcels, as shown in the diagram below:

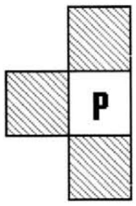


a)

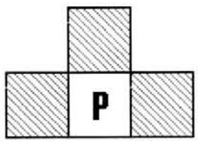
Diagrams *b)*, *c)* and *d)* are obtained by rotating the diagram *a)* about its center through the angles of 90, 180 and 270 degrees respectively.



b)



c)



d)

According to the theorem, all of these arrangements produce the same number of weed parcels and it is thus enough to examine only one of them. Here, the case *a)* will be examined.

Because of the way in which it is surrounded by the weed parcels, it is obvious that parcel **P** cannot be in the first row, last row or last column of the field. However, no matter which

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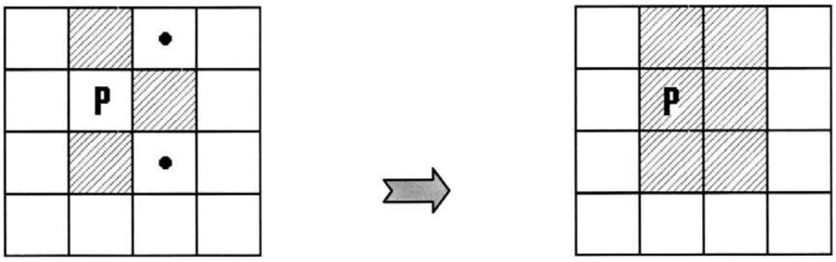
of the remaining parcels is chosen to be the parcel **P**, only that parcel and two more parcels will grow into weed. Therefore, with this kind of arrangement of the weed parcels there will always be 10 weed parcels which can not grow into weed, so it is impossible for the whole field to grow into weed.

One of the possibilities is shown in the diagram below. In this case only the parcel **P** and the two dotted parcels will grow into weed after some time.



2nd case: Parcel **P** is adjacent to two weed parcels

The two possibilities for arranging the weed parcels around the parcel **P** in this case are the same as the two possibilities already analyzed in case of having a field of dimensions 3*m* x 3*m*.

a)

b)

Considering these two possibilities I will try to place the third weed parcel in the field in such a way so that the whole field grows into weed. The two arrangements equivalent to the arrangements *a)* and *b)* will not be further analyzed.

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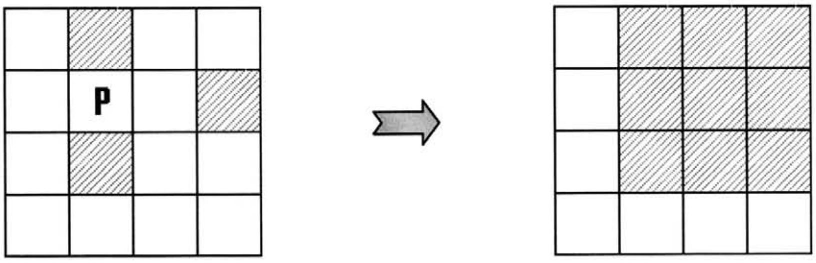
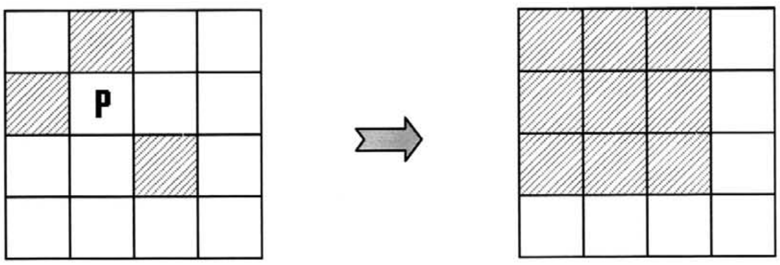
If the parcel **P** is adjacent to the two weed parcels as on the diagram *a)*, then one of the ways of placing the third weed parcel so that the maximum number of parcels turns into weed is shown on the diagram below. In every other case the final number of weed parcels will be less, and consequently there will be more fields which will never grow into weed.



It is easy to see that in this arrangement the maximum number of parcels which will turn into weed is 9. Therefore, there are 4 x 4 – 9 = 7 weed parcels which can never grow into weed, so in this case the whole field can never grow into weed.

The similar conclusion is obtained if the parcel **P** and the adjacent weed parcels are arranged as on the diagram *b)*. If the third weed parcel is placed so that the maximal number of weed parcels is obtained, there will remain 7 parcels which will never grow into weed. This implies that neither in this case can the whole field completely grow into weed.

One of the ways of placing the third parcel so that we get the maximal number of weed parcels is shown on the diagram below.



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Therefore, by using the system of exhaustion, it has been proved that *the field with dimensions 4m x 4m and 3 weed parcels in it can never grow into weed.* The maximum number of weed parcels is produced in the 2[nd] case and it is 9. Therefore, at least 7 parcels in this field will never grow into weed.

Hypothesis

Even though the system of exhaustion is not the proper way to prove or disprove the problem, by applying it to the simpler variations of the main problem it is possible to set a hypothesis. Namely, since in all of the previously analyzed examples it was impossible to find an arrangement of the weed parcels so that the whole field grows into weed, it is possible to conjecture that there will be no solution in case of having a field with dimensions 10*m* x 10*m* and 9 weed parcels. Also, it is possible to generalize this problem and conjecture even further that a field of dimensions n x n with n-1 initial weed parcels can not become fully overgrown by weed.

The table below shows the number of initial weed parcels and the maximum number of weed parcels for the fields in each of the analyzed cases.

Dimensions (n)	Number of weed parcels (n-1)	Max number of weed parcels (*max*)
1*m* x 1*m*	0	0
2*m* x 2*m*	1	1
3*m* x 3*m*	2	4
4*m* x 4*m*	3	9

We can observe that each of the cases $max = (n-1)^2$.
Therefore, it is possible to conjecture that in the general case the maximum number of the parcels which will grow into weed is equal to the square of the number of initial weed parcels.

Proving the hypothesis

The system I will use to prove this hypothesis will involve the following of a change in a certain variable. More precisely, I will follow the change of the total perimeter of the areas covered with weed.

If **P** is a randomly chosen non-weed parcel within the field, we can note that there are three situations when it will grow into weed. Obviously, if parcel **P** is not adjacent to at least two weed parcels, it can never become a weed parcel itself. Therefore, those three cases are when:

- parcel **P** is adjacent to 2 weed parcels;
- parcel **P** is adjacent to 3 weed parcels; and
- parcel **P** is adjacent to 4 weed parcels.

In the further study of the problem I will analyze how the total perimeter of the weed-areas changes in each of the cases. However, the cases which can be obtained by rotating the field will not be examined because they will produce the same number of new weed parcels. Furthermore, I will only follow the change in the perimeter of the weed-areas in the field section around a parcel **P**, without focusing on the position of the parcel **P** in the field. In doing so, I will consider that the total perimeter of the remaining weed-areas is unchanged. This way I will be able to conclude how the total perimeter of all weed-areas changes in each case, as the change in the perimeter in one section of the field affects the change of the perimeter in the whole field.

1st case: Parcel **P** is adjacent to 2 weed parcels

The diagram of what will happen after some time is shown below.

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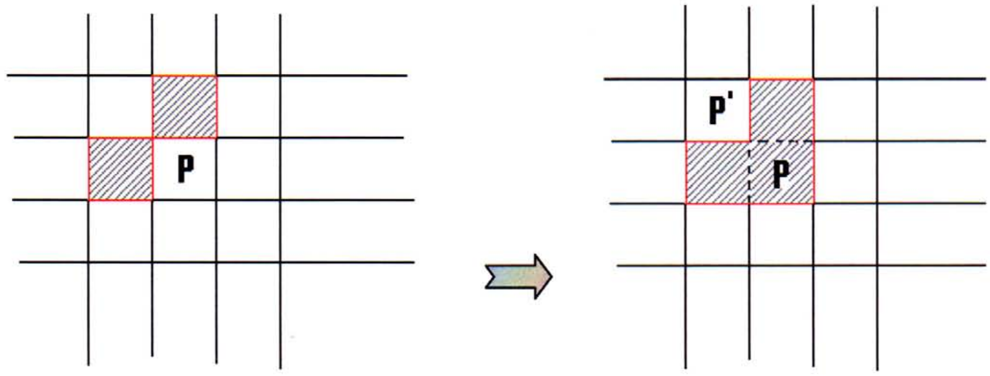
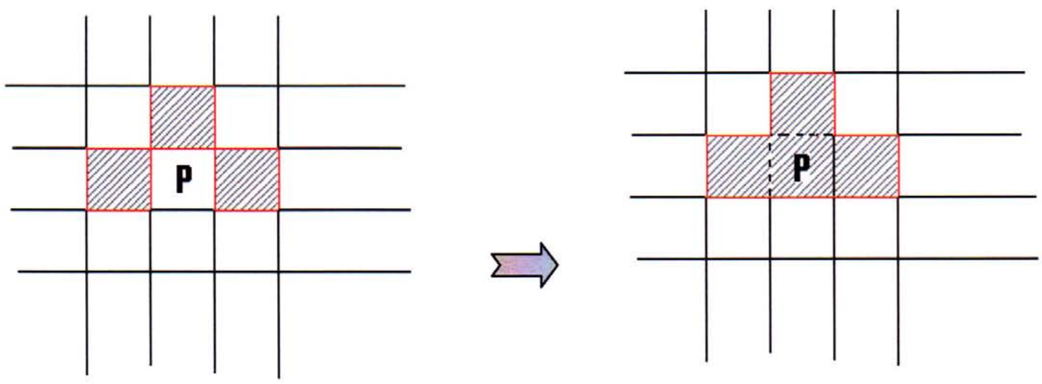


Since each parcel in the field has all sides of length 1*m,* it is easy to calculate the total perimeter of the weed parcels.
Looking at the diagram we can note that the perimeter of the weed-areas in this section is initially 2 x 4*m* = 8*m*, and after the parcel **P** grows into weed the perimeter is also 8*m*.
So, if the parcel **P** is adjacent to two weed parcels, the perimeter of the weed-areas does not change. Hence, the total perimeter of all weed-areas will also not change.

The parcel **P'** is also adjacent to 2 of the weed parcels, but even after it grows into weed the perimeter will still remain unchanged.

2nd case: Parcel **P** is adjacent to 3 weed parcels

One of the possible arrangements is shown on the following diagram



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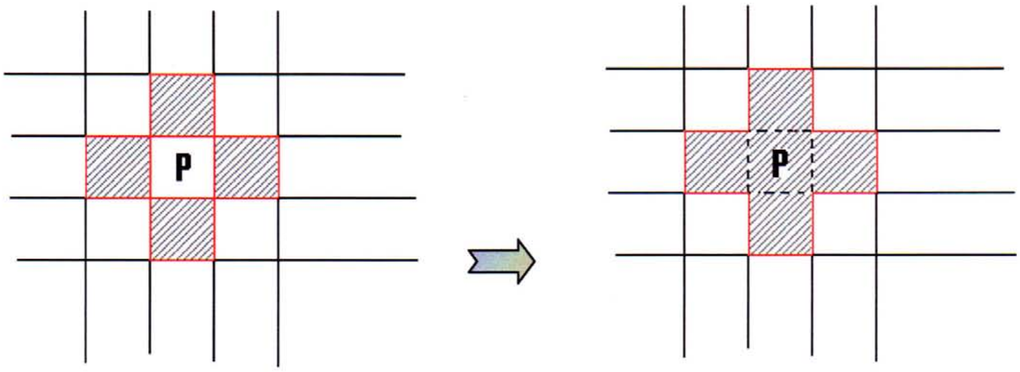
Before the parcel **P** grows into weed the perimeter of the weed-areas in this section is 3 x 4m = 12m. However, after **P** becomes a weed parcel, the perimeter of the weed-area is 10m. So, if **P** is adjacent to 3 weed parcels, after it turns into weed the total perimeter in this field section decreases by 2. Thus, the total perimeter of all weed-areas in the field decreases by 2.

3rd case: Parcel **P** is adjacent to 4 weed parcels

The case is shown in the diagram below:



The perimeter of the weed-areas in this section before **P** becomes a weed parcel is 4 x 4m = 16m, and after **P** becomes overgrown by weed the perimeter is 4 x 3m = 12m. So, if **P** is initially adjacent to 4 weed parcels, the total perimeter of weed-areas in this section decreases by 4 after some time. Therefore, the total perimeter of all weed-areas in the field decreases by 4.

Conclusion 1

Analyzing each of the possible cases, we can conclude that no matter how the weed parcels are initially placed in the field, *the total perimeter of the weed parcels either stays the same or reduces.*

Since the initial total perimeter of all weed-areas is 9 x 4*m* = 36*m*, and the perimeter of the field is 4 x 10*m* = 40*m*, *it is impossible to place the weed parcels in such a way that the whole field grows into weed.*

However, since a field of dimensions 9*m* x 9*m* has the perimeter of 36*m*, it can grow into weed if the weed parcels in the field are strategically arranged. So the maximum number of weed parcels obtained by strategically arranging the 9 initial weed parcels is 9^2 (the number of initial weed parcels squared).

Field with dimensions *n* x *n* and *(n-1)* weed parcels

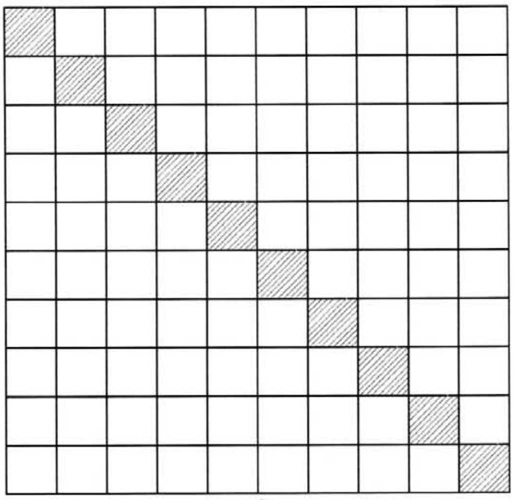
By following the change of the perimeter in the same way as above it is possible to prove that a field with dimensions *n* x *n* and *n-1* initial weed fields can never grow into weed. Also, using the same method and the already obtained results, we have proved that the minimal number of initial weed parcels needed for the field of dimensions *n* x *n* to turn into weed is *n*. If those *n* parcels are strategically arranged within the field, they will infect the whole field.

Extending the initial problem

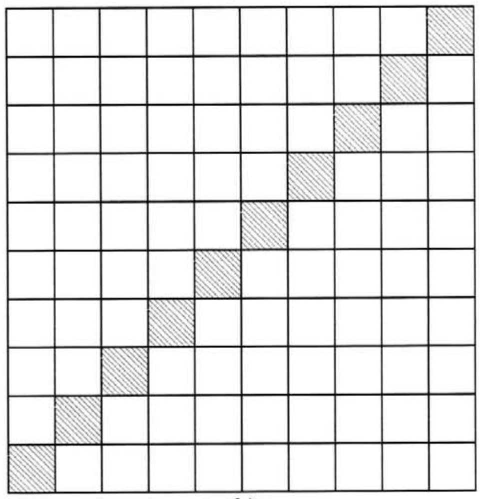
Knowing that n weed parcels are needed in order to turn a field of dimensions n x n into weed, the initial problem with the field of dimensions 10*m* x 10*m* can now be extended with the following question:

Find all the configurations of placing 10 weed parcels into the field of dimensions 10m x 10m so that the whole field grows into weed after some time.

The most obvious way to place the 10 weed parcels so that the whole field grows into weed is placing the weed parcels on the diagonal of the field.

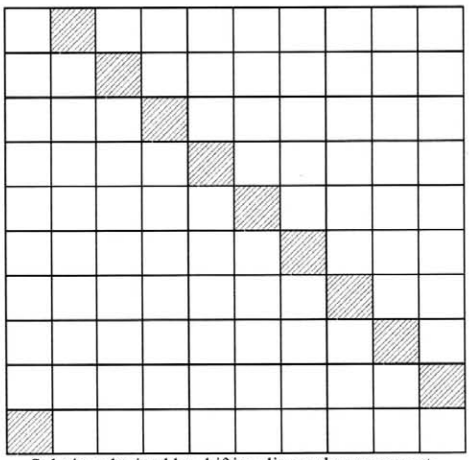


a)

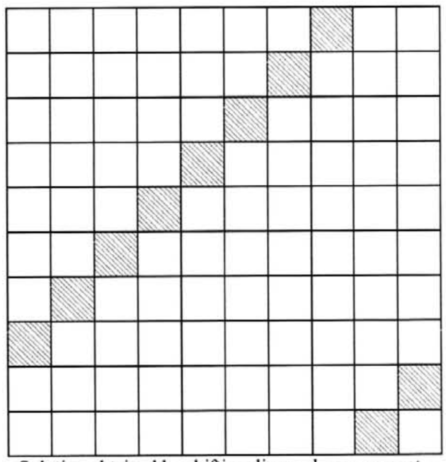


b)

By shifting the diagonal arrangements to the left or to the right, some other trivial solutions can be obtained. Two of the solutions obtained in this way are shown on the diagram below.



Solution obtained by shifting diagonal arrangement on diagram a) for 1 place to the right



Solution obtained by shifting diagonal arrangement on diagram b) for 2 places to the left

Since there are ${}^{100}C_{10} = \dfrac{100!}{90!*10!} = 17310309456440$ possibilities to arrange 10 weed-parcels in the field with $10 \times 10 = 100$ parcels, rather than drawing every possible arrangement of the weed parcels and checking whether it is valid, I will try to develop a system for finding all the possible solutions.

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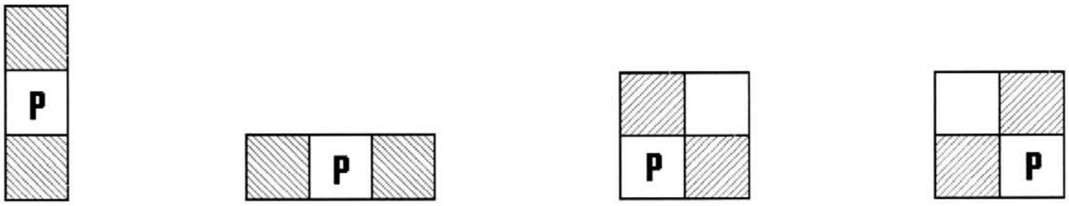
It was shown earlier that ***only*** if a random parcel **P** has initially only 2 adjacent weed-parcels, the total perimeter of the weed parcels will not change. The total perimeter of the field is: $4 \times 10m = 40m$, and the total perimeter of all 10 weed parcels is initially $10 \times 4m = 40m$. Therefore, the total perimeter must not reduce. This means that at any moment there must not be any non-weed parcels **P** with more than 2 adjacent weed parcels.

So, a randomly chosen non-weed parcel **P** can be in one of the following arrangements:



If we number the weed parcels with **1, 2,…,10** and if we consider the field as a matrices **F(10 x 10)**, we can introduce the position of every weed parcel by (i_n, j_n) where ***n*** is the number of the parcel, ***i*** is the row and ***j*** is the column of the matrices **F**.

From this point I will refer to rows **1** and **10** and columns **1** and **10** as *borders*, since the bordering edges of the field are contained in them.

It is trivial to conclude that if there is not a weed parcel in a border even if all the other parcels had weed, the border would not grow into weed. So in order for the whole field to grow into weed, there must initially be weed parcels in all of the 4 borders. However, this does not necessarily mean that four different bordering parcels have to be used. For example, a parcel placed at corner (10, 1) is at the same time in two borders: row 10 and column 1.

Using this knowledge about the borders and the initial weed parcels, I will develop an algorithm for finding all the possible arrangements of 10 weed parcels which will lead to the whole field growing into weed.

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The principle I will use for this is the principle of recursion and backtracking. Namely, I will assume that all of the initial weed parcels are already placed in the field so that the whole field is overgrown by weed. Then, by considering the possible positions of the last weed parcel and the dependence of the remaining weed parcels on it, I will try to find all the possible positions of the parcel which was placed the 9^{th}, then analogically all the possible positions of the parcel that was placed the 8^{th}, etc.

Since each border should contain a weed parcel, I will consider one of the parcels which go into a border as parcel **10**(the parcel which was put in the field the last). Since it does not matter which border parcel **10** belongs to (by using recursion and backtracking all the possibilities will systematically be exhausted) in the following analyses I will consider parcel **10** to be a parcel placed in the 10^{th} column.

Depending on whether the parcel **10** is placed in the corner (10,10) or in one of the positions (i,10), where$1 \le i \le 9$, there will be 2 different cases. However, before analyzing these cases any further, I will introduce a theorem and its consequence.

Theorem: If all initial weed parcels are strategically placed in the field and there are no more non-weed parcels which can turn into weed parcels, the shape of the area which is overgrown by weed is a rectangle.

proof: Since the aim is to cover the whole field with weed, the weed parcels will be placed so that they create only one weed area in the field. It is obvious that in other cases the field cannot grow into weed.
Let's assume the proposition opposite of the theorem is true: there are no more parcels which can grow into weed and the shape of the area which has grown into it is not a rectangle.
If the weed area does not have a rectangular shape, somewhere in the field there must be a non-weed parcel **P** adjacent to at least 2 weed parcels (one of the possibilities is shown on the diagram below, other possibilities can be obtained by rotating it through different angles).

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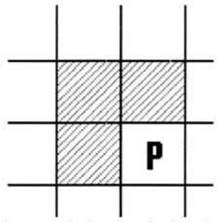


But, according to the condition of the original problem, the parcel **P** *will* grow into weed. This leads to a contradiction, so the area covered with weed must have the shape of a rectangle.

Consequence: If $\mathcal{A}$ is the area of the rectangle covered with weed, it can be calculated using the equation:

$$\mathcal{A} = (i_{max}-i_{min}+1) \times (j_{max}-j_{min}+1),$$

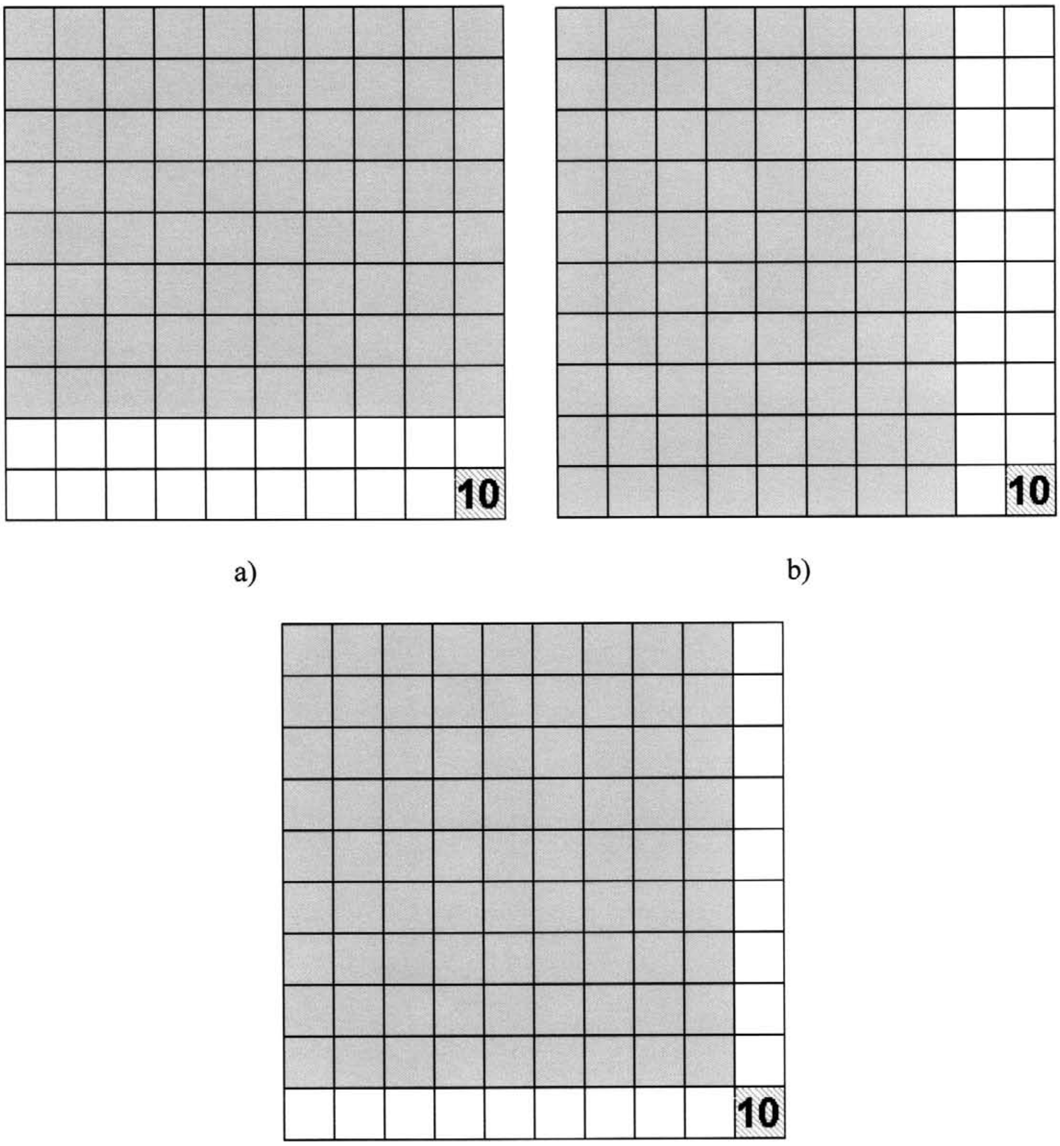
where $i_{max}= \max(i_1,\ldots,i_n)$, $j_{max}= \max(j_1,\ldots,j_n)$, $i_{min}= \min(i_1,\ldots,i_n)$, $j_{min}= \min(j_1,\ldots,j_n)$ and n is the number of initial weed parcels in the field.

proof: In case of n=1, $i_{min}=i_{max}$ and $j_{min}=j_{max}$ so $\mathcal{A} = (i_{max}-i_{min}+1) \times (j_{max}-j_{min}+1)=1$. So, for n=1 the equation is true. For n>1 the proof can easily be seen if the corresponding diagrams are drawn.

Now I will further develop the two cases depending on the position of the parcel **10**.

1^st^ case: Parcel **10** is in the corner

Since there must be some weed parcels in row 1 and column 1, $i_{min}=1$ and $j_{min}=1$.
In order for all of the parcels to grow into weed, the parcel **10** must be in a relation with another weed parcel like the two shaded parcels are related in the diagrams on the page 14 of the essay. Since the position of the parcel **10** is $(i_{10},j_{10})=(10,10)$, there must be a weed parcel at the position $(i_x,j_x)=\{ (8,10), (10,8), (9,9) \}$. This automatically gives the values of i_{max} and j_{max}, since $i_{max}=i_x$ and $j_{max}=j_x$, and thus determines the shape of the rectangles. So, if the parcel 10 is in the corner, the remaining 9 parcels have to form the required rectangle in each case. All of the possible cases are shown on the diagrams below.



a) b) c)

The perimeter of the shaded area (the area overgrown by weed) in each of the diagrams *a)*, *b)* and *c)* is 36*m*. The total perimeter of the 9 initial weed parcels which belong to these areas is also 36*m*. Therefore, it is possible to produce the shaded areas with those 9 weed parcels.

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Now finding all the possible arrangements for this case comes to finding all the arrangements for placing 9 weed parcels into fields of dimensions 10*m* x 8*m*, 8*m* x 10*m* or 9*m* x 9*m* so that they are overgrown by weed. These arrangements can be found by the system of recursion, using the similar analyses and reducing the number of weed fields and the area they have to cover. By the system of backtracking, when the shaded area is reduced to its minimum, all the possible arrangements when the parcel **10** is at the position (10, 10) can be drawn.

2nd possibility: parcel **10** is in one of the positions (i,10), where $1 \le i \le 9$

Since some of the remaining initial weed parcels should be in each of the remaining borders, $i_{min}=1$ and $j_{min}=1$. I will refer to the parcel in the row 10 as to parcel **9**.
In order for the whole field to grow into weed, the parcels **1-8** have to be arranged in a way which produces the maximal number of weed parcels.

The total perimeter of the remaining 8 parcels is 8 x 4*m* = 32*m*. Therefore, the area covered by weed as a result of arranging these 8 parcels can have the maximal perimeter of 32*m*. Since that area will have the shape of a rectangle of dimensions a and b, its perimeter is $2\times(a+b)=32 \Rightarrow a+b=16$. So, the possibilities for this are: (a, b) = {(10, 6), (9, 7), (8, 8), (7, 9), (6, 10)}.

All 5 possibilities are shown on the following diagrams:

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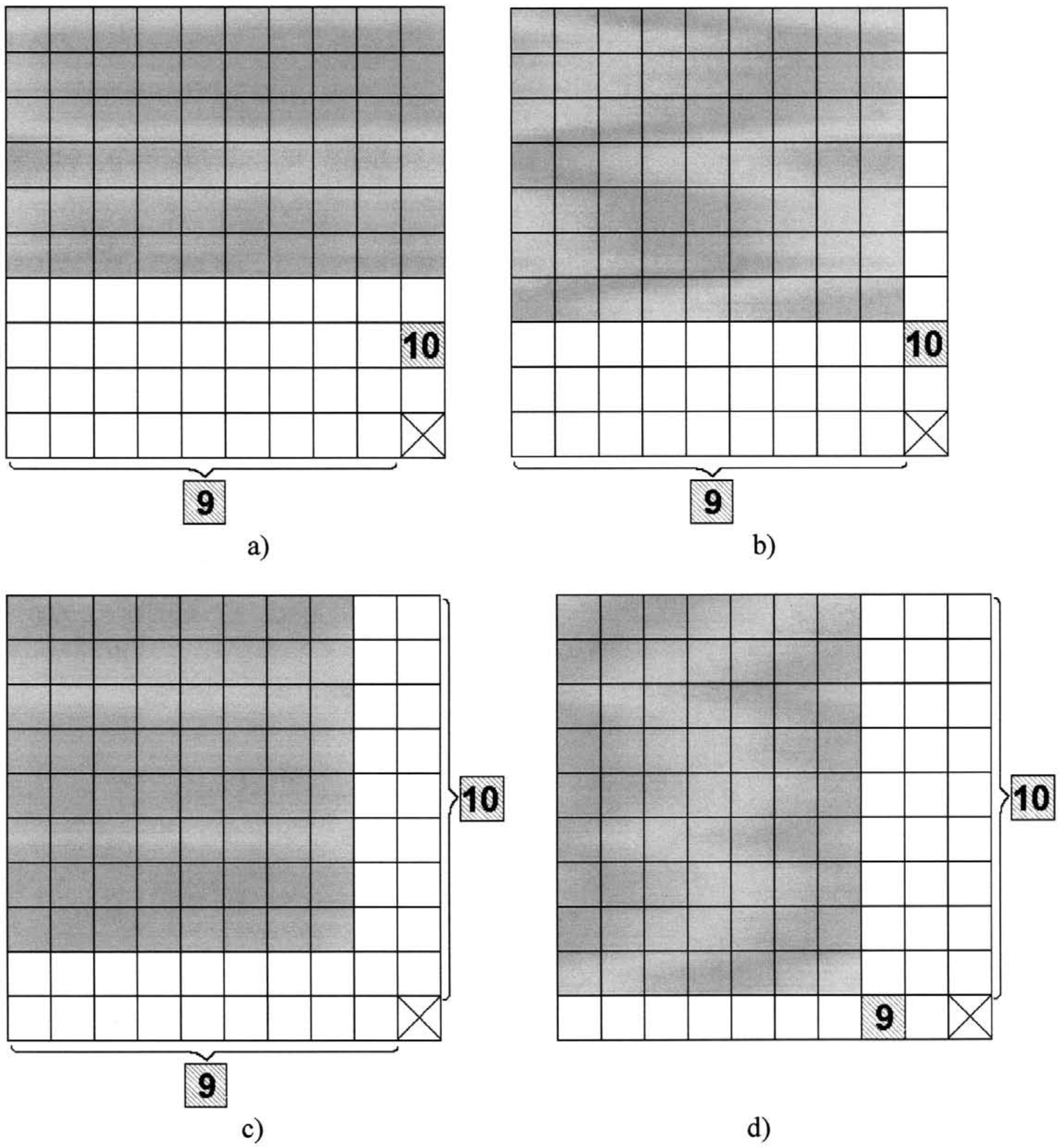

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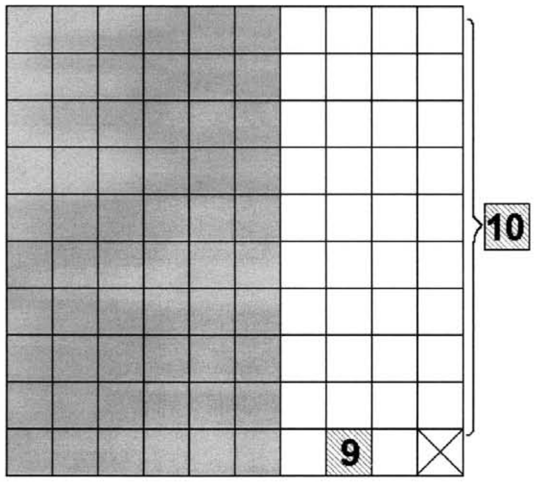

e)

In the first case there is only one possibility for placing the parcel **10**. In any other case it would be impossible for the whole field to grow into weed. There are 9 possibilities for placing parcel **9**. Altogether, the number of the possibilities in this case is 1 x 9 x (number of possibilities to obtain the rectangle of dimensions 10 x 8). All the arrangements for obtaining the rectangle can be found by recursion, by applying the same analyses as for the field of dimensions 10 x 10.

In case *b)* there is also only one possibility for placing the parcel **10**, and 9 possibilities for placing the parcel **9**. The arrangement of other 8 parcels which will form the rectangle can be found by using the similar method as for field 10 x 10. Therefore, all the possible arrangements in this case can be drawn by applying the recursion and backtracking.

It is a similar situation with the following 3 cases. The only differences are that in case *c)* there are 9 possibilities for placing each of the parcels **9** and **10** and that cases *d)* and *e)* are symmetrical to the cases *a)* and *b)*, now with one possibility for parcel **9** instead of parcel **10**.

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Conclusion 2

Therefore, by fully analyzing these two possibilities, it was shown that by the principle of recursion and backtracking all the possible arrangements would be exhausted. Thus, this principle would be a very good one to use if we wanted to draw all the possible arrangements of the 10 initial weed parcels which would produce a complete weed field, since it would not produce any identical arrangements.

This system for finding the solutions can easily be generalized for a field with dimensions n x n and n initial weed parcels.

References:

-Web-site: **http://www.matf.bg.ac.yu/~matic/competitions/dodatne/ZKOMB_mll.pdf**

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